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GAMES ARISING FROM INFINITE PRODUCTION SITUATIONS

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Situations involving the linear transformation of products (LTP) with an infinite number of transformation techniques are considered. It is shown that an optimal solution of the dual program of the profit "maximisation" problem exists under certain conditions. Moreover, this solution generates a core-element of the corresponding LTP game.

1. Introduction

Owen (1975) introduced linear production (LP) situations. These are production situations where there is a finite set of producers, each of them owns a bundle of resources and all producers can use the same finite set of linear production techniques. The products can be sold on the market at given prices and all producers are price takers. This model has two restrictions. First, each production process can only have one good output while in practice many production processes have by-products. Second, all producers can use the same production techniques while in reality some producers have a production technique that nobody else has. To overcome these restrictions, Timmer, Borm and Suijs (2000) introduced situations involving the linear transformation of products (LTP). In these situations, each linear transformation technique has at least one good output and different producers may have different production techniques. More precisely, in an LTP situation there is a finite set of producers and each of them controls a finite number of transformation techniques. We define the set of goods to be the set of products and resources. Each producer owns a bundle of goods that he can use (like resources) in his transformation process or that he can sell directly on the market (like products). The outcome of the transformation process, the produced goods, will also be sold on the market. The goal of each producer is to maximise his profit given his transformation techniques, bundle of goods and the exogenous market prices.

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In this paper, we consider semi-infinite LTP situations, which are LTP situations with a countable, infinite number of transformation techniques. Something similar for LP situations and corresponding games has been studied by Fragnelli, Patrone, Sideri and Tijs (1999) using duality results by Tijs (1979). We will also work with linear semi-infinite programs. One of the first papers in this area was written by Charnes, Cooper and Kortanek (1962). Many results on this subject can be found in Glashoff and Gustafson (1983) and in the recent book by Goberna and López (1998).

This paper is organised as follows. Section 2 starts with an introduction to semi-infinite LTP situations where we consider a countable, infinite number of transformation techniques. Some examples show which problems we may encounter. Therefore, in each of the Secs. 3 and 4, a set of conditions will be presented that ensures the existence of an optimal dual solution and the existence of a core-element of the corresponding semi-infinite LTP game.

2. Semi-Infinite LTP Situations

In many production situations, there are an infinite number of techniques available to the producer. For example, a firm may have a finite number of transformation techniques on the short run, but when we think of the long run, this firm can choose from an infinite number of techniques. It can continue its current production process, it can expand its activities, it can produce some extra goods or it can switch to the use of some completely different transformation techniques. A second example concerns cooking. If you have a recipe for baking pancakes from flour, milk, eggs, butter and sugar, then you can get an infinite number of recipes for pancakes by changing the amounts of the ingredients slightly. Each recipe then gives a slightly different pancake.

A semi-infinite LTP situation models such a production situation with a countably infinite number of transformation techniques. It is denoted by a five-tuple $\langle N, A, D, \omega, p\rangle$ where N is the finite set of producers, $A \in \mathbb{R}^{M\times D}$ is the technology matrix where each column represents a transformation technique, M is the finite set of goods and D is a countably infinite set of transformation techniques. We assume that each technique needs at least one input good to produce at least one good output. The resource bundle of producer $i \in N$ is $\omega(i) \in \mathbb{R}^M_+$, $\omega = (\omega(i))_{i\in N}$ and $p \in \mathbb{R}^M_+ \setminus \{0\}$ denotes the exogenous market prices. Define the resource matrix $G \in \mathbb{R}^{M\times D}$ by $G_{jk} = \max\{0, -A_{jk}\}$ and $\omega(S) = \sum_{i\in S}\omega(i)$, $S \subset N$. The set $D(i)$ contains all the transformation techniques controlled by producer i. Then $D(S) = \cup_{i\in S} D(i)$ are the techniques available to coalition $S \subset N$ and $D = D(N)$. The activity level of technique $k \in D$ is denoted by $y_k \geq 0$. The corresponding LTP game (N, v) is such that coalition S receives the smallest upper bound of its profit,

$$\begin{array}{rll} v(S) = \sup & p^T(\omega(S) + Ay) & \\ \text{s.t.} & Gy \leq \omega(S) & \\ & y_k = 0 \qquad \text{if} \quad k \notin D(S) & \\ & y \geq 0\,. & \end{array}$$

The core $C(v)$ of a game (N, v),

$$C(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \quad \text{for all} \quad S \subset N \right\},$$

is the set of allocations x of $v(N)$ upon which no coalition S of producers can improve. A game is called *balanced* if it has a non-empty core and it is called *totally balanced* if each subgame $(S, v_{|S})$ has a non-empty core, where $v_{|S}(T) = v(T)$ for all $T \subset S$. The following examples show some problems we may encounter in semi-infinite LTP situations.

Example 2.1. Consider the semi-infinite LTP situation with a single producer, two goods, bundle of goods $\omega = (3, 0)^T$, market prices $p = (1, 3)^T$ and technology matrix

$$A = \begin{bmatrix} -4 & -3\frac{1}{2} & -3\frac{1}{3} & -3\frac{1}{4} & \cdots & -3 - \frac{1}{k} & \cdots \\ 2 & 2 & 2 & 2 & \cdots & 2 & \cdots \end{bmatrix}.$$

The kth column a^k of A represents transformation technique $k \in D$. Negative numbers in this column indicate input goods of the technique whereas positive numbers indicate goods output. The primal profit "maximisation" problem is

$$\begin{aligned} \sup \quad & p^T(\omega + Ay) \\ \text{s.t.} \quad & Gy \leq \omega \\ & y \geq 0 \,. \end{aligned}$$

Note that we use the supremum since there are an infinite number of activity levels y_k, $k \in D$, and an optimal solution may not exist. This problem is equal to

$$\sup \left\{ 3 + p^T Ay \,\middle|\, \sum_{k=1}^{\infty} (3 + 1/k) y_k \leq 3, \ y \geq 0 \right\} = \lim_{k \to \infty} (3 + 3 - 1/k) = 6 \,.$$

There is no optimal solution for this problem, that is, there exists no vector $\hat{y}$ of activity levels such that $p^T(\omega + A\hat{y}) = 6$. The corresponding dual problem is

$$\begin{aligned} &\inf\{(z+p)^T \omega | G^T z \geq A^T p, \ z \geq 0\} \\ &\quad = \inf\{3z_1 + 3 \,|(3 + 1/k) z_1 \geq 3 - 1/k, \ k = 1, 2, \ldots, z \geq 0\} \\ &\quad = 3 \cdot 1 + 3 = 6 \,. \end{aligned}$$

The set of optimal solutions $\{z \in \mathbb{R}^2 | z_1 = 1, z_2 \geq 0\}$ is non-empty. In this example, we see that the primal problem may have no optimal solution while the dual problem has optimal solutions.

Example 2.2. Consider the following semi-infinite LTP situation with a single producer, two goods, bundle of goods $\omega = (0, 1)^T$, prices $p = (1, 1)^T$ and technology matrix

$$A = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & \cdots & -\frac{1}{k} & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 1 & \cdots \end{bmatrix}.$$

Then

$$\begin{aligned} v &= \sup\{p^T(\omega + Ay) \mid Gy \leq \omega,\ y \geq 0\} \\ &= \sup\left\{1 + p^T Ay \,\middle|\, \sum_{k=1}^{\infty} y_k/k \leq 0,\ y \geq 0\right\} = 1 + 0 = 1 \end{aligned}$$

with optimal activity vector $y = 0$. The dual problem equals

$$\begin{aligned} &\inf\{z_2 + 1 \mid z_1/k \geq 1 - 1/k,\ k = 1, 2, \ldots,\ z \geq 0\} \\ &= \inf\{z_2 + 1 \mid z_1 \geq k - 1,\ k = 1, 2, \ldots,\ z \geq 0\} = +\infty \end{aligned}$$

since there exists no feasible solution z. Therefore there are no optimal solutions to the dual program of this example while there exists an optimal solution to the primal problem.

Example 2.3. We have a semi-infinite LTP situation with a single producer, five goods and

$$A = \begin{bmatrix} -2 & -2 & -2 & -2 & & -2 & \\ 0 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & & -\frac{1}{k} & \\ 1 & 0 & 0 & 0 & \cdots & 0 & \cdots \\ 1 & 1 & 1 & 1 & & 1 & \\ 0 & 1 & 1 & 1 & & 1 & \end{bmatrix}, \quad \omega = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad p = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \\ 4 \end{bmatrix}.$$

The profit maximisation problem gives

$$v = \sup\{2 + p^T Ay \mid 0 \leq y_1 \leq 1,\ y_k = 0,\ k = 2, 3, \ldots\} = 4\,.$$

The corresponding dual problem gives

$$\inf\{2 + 2z_1 \mid 2z_1 \geq 2,\ 2z_1 + z_2/k \geq 3,\ k = 2, 3, \ldots, z \geq 0\} = 5\,.$$

Here we have a duality gap: the primal maximisation program does not have the same optimal value as the dual problem.

These examples show that semi-infinite LTP situations may deal with duality gaps and the absence of optimal solutions for both the primal and the dual program. We would like to have conditions on semi-infinite LTP situations such that there is no duality gap and the dual problem has an optimal solution. Then we can easily find a core-element of the game via the dual problem (we show this in the next section). We do not need the existence of an optimal solution of the primal problem to attain this core-element.

In the following two sections, we present two sets of conditions that ensure we can find a core-element of the LTP game corresponding to a semi-infinite LTP situation via the dual problem.

3. Conditions Involving Cones

In this section, we will present a first set of conditions on semi-infinite LTP situations and we show that this guarantees that the corresponding LTP games have a non-empty core.

Denote by 0_M the M-dimensional zero-vector, and by e^j the jth unit vector in $\mathbb{R}^M$ with $e^j_m = 1$ if $m = j$ and $e^j_m = 0$ if $m \neq j$. If B is an (infinite) set of vectors in $\mathbb{R}^q$ for some integer number q, then we obtain the convex cone generated by B, denoted by $CC(B)$, by taking all non-negative multiples of finite convex combinations of elements in B. Thus,

$$CC(B) = \left\{ x \,\middle|\, x = \sum_{i=1}^{t} \lambda_i b^i,\ \lambda_i \geq 0,\ b^i \in B,\ i = 1, 2, \ldots, t,\ t \geq 1 \right\}.$$

Define the sets K_1 and K_2 as follows:

$$K_1 = CC\left(\{(g^k)_{k\in D},\ (e^j)_{j\in M}\}\right) = \mathbb{R}^M_+$$

$$K_2 = CC\left(\left\{ \begin{pmatrix} g^k \\ p^T a^k \end{pmatrix}_{k\in D},\ \begin{pmatrix} e^j \\ 0 \end{pmatrix}_{j\in M} \right\}\right).$$

The last equality for K_1 follows from $g^k \in \mathbb{R}^M_+$ for all $k \in D$. In the literature, see for example Glashoff and Gustafson (1983) and Goberna and López (1998), the convex cones K_1 and K_2 are usually called the first and second moment cone and denoted by M and N respectively. We renamed these cones since we already use M and N to denote respectively the set of goods and the set of producers. Denote by $\mathrm{int}(K_1)$ the interior of K_1 and by $cl(K_2)$ the closure of K_2. Consider the following two conditions.

Condition 3.1. $\omega(N) \in \mathrm{int}(K_1) = \mathbb{R}^M_{++}$

This condition states that the coalition N of all producers should own some positive amount of all goods in M.

Condition 3.2. $\binom{0_M}{1} \notin cl(K_2)$

An interpretation of this condition is that doing nothing (which is equivalent to activity level $y_k = 0$ for all $k \in D$) cannot result in a positive profit. The following theorem shows the non-emptiness of the core under these conditions.

Theorem 3.1. *Let $\langle N, A, D, \omega, p\rangle$ be a semi-infinite LTP situation. If Conditions* 3.1 *and* 3.2 *are satisfied, then the corresponding LTP game is balanced.*

Proof. Conditions 3.1 and 3.2 are satisfied and therefore it follows from respectively Theorems 8.1(v), (vi) and 4.4(i) in Goberna and López (1998) that the dual problem for coalition N,

$$\begin{array}{ll} \inf & (z+p)^T\omega(N) \\ \text{s.t.} & G^Tz \geq A^Tp \\ & z \geq 0\,, \end{array}$$

is feasible, there exists an optimal dual solution and there is no duality gap. Let $\underline{z}$ be an optimal dual solution. Define $x \in \mathbb{R}^N$ by $x_i = (\underline{z}+p)^T\omega(i)$ for all $i \in N$. Then $\sum_{i\in N} x_i = \sum_{i\in N}(\underline{z}+p)^T\omega(i) = (\underline{z}+p)^T\omega(N) = v(N)$ where the last equality holds because there is no duality gap. Notice that $\underline{z}$ is also a feasible solution of the problem $\inf\{(z+p)^T\omega(S)|G^Tz \geq A^Tp,\ z \geq 0\}$ for all coalitions S. Therefore,

$$\begin{aligned} (\underline{z}+p)^T\omega(S) &\geq \inf\{(z+p)^T\omega(S)|G^Tz \geq A^Tp,\ z \geq 0\} \\ &= \sup\{p^T(\omega(S)+Ay)|Gy \leq \omega(S),\ y \geq 0\} \\ &\geq \sup\{p^T(\omega(S)+Ay)|Gy \leq \omega(S),\ y_k = 0 \text{ if } k \notin D(S),\ y \geq 0\} \\ &= v(S) \end{aligned}$$

and $\sum_{i\in S} x_i = \sum_{i\in S}(\underline{z}+p)^T\omega(i) = (\underline{z}+p)^T\omega(S) \geq v(S)$. We conclude that $x \in C(v)$. □

We will now return to our examples in the previous section. In the first example we have that $g^k = (3+\frac{1}{k}, 0)^T$ and $p^Ta^k = 3 - \frac{1}{k}$ for all $k \in D$. Thus

$$K_1 = CC\left\{\left(\begin{matrix} 3+\dfrac{1}{k} \\ 0 \end{matrix}\right)_{k\in D},\ \left(\begin{matrix} 1 \\ 0 \end{matrix}\right),\ \left(\begin{matrix} 0 \\ 1 \end{matrix}\right)\right\} = \mathbb{R}^2_+$$

and

$$K_2 = CC\left\{\left(\begin{matrix} 3+\dfrac{1}{k} \\ 0 \\ 3-\dfrac{1}{k} \end{matrix}\right)_{k\in D},\ \left(\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}\right),\ \left(\begin{matrix} 0 \\ 1 \\ 0 \end{matrix}\right)\right\}.$$

We see that Condition 3.2 is satisfied since $(0,0,1)^T \notin cl(K_2)$, but Condition 3.1 is not satisfied because $\omega_2 = 0$. However, there is no duality gap and there exists an optimal dual solution.

In the second example, we see that $g^k = (\frac{1}{k}, 0)^T$ and $p^T a^k = 1 - \frac{1}{k}$ for all $k \in D$. Therefore $K_1 = \mathbb{R}^2_+$ and

$$K_2 = CC\left\{\begin{pmatrix} \frac{1}{k} \\ 0 \\ 1-\frac{1}{k} \end{pmatrix}_{k\in D}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}.$$

Here, Condition 3.1 is not satisfied since $\omega_1 = 0$ and the same holds for Condition 3.2 since $(0,0,1)^T \in cl(K_2)$. The dual problem has no feasible solutions.

Finally, in the third example we have that $g^1 = (2,0,0,0,0)^T$, $g^k = (2, \frac{1}{k}, 0, 0, 0)^T$, $k \geq 2$, $p^T a^1 = 2$ and $p^T a^k = 3$, $k \geq 2$. So $K_1 = \mathbb{R}^5_+$ and

$$K_2 = CC\left\{\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ \frac{1}{k} \\ 0 \\ 0 \\ 0 \\ 3 \end{pmatrix}_{k\geq 2}, \begin{pmatrix} e^j \\ 0 \end{pmatrix}_{j\in M}\right\}.$$

In this example, Condition 3.1 is not satisfied but Condition 3.2 is and there is a duality gap.

From these examples, we may conclude that Conditions 3.1 and 3.2 are sufficient but not necessary conditions in Theorem 3.1.

4. Economic Conditions

In this section, a second set of conditions on semi-infinite LTP situations will be presented. These conditions guarantee total balancedness of the corresponding LTP games. Similar conditions for linear production (LP) situations can be found in Fragnelli, Patrone, Sideri and Tijs (1999).

Condition 4.1. $\sup_{k\in D} p^T a^k = \gamma < +\infty$

All transformation techniques a^k should generate a finite profit of at most γ when $y_k = 1$, that is, the techniques are operated at the unit activity level.

Condition 4.2. $\max_{j\in M} g^k_j \geq \alpha > 0$ for all $k \in D$

This condition states that for each transformation technique, there is always some positive amount α of a resource needed at the unit activity level.

Recall that

$$\begin{aligned} v(N) = \sup \quad & p^T(\omega(N) + Ay) \\ \text{s.t.} \quad & Gy \leq \omega(N) \\ & y \geq 0\,. \end{aligned}$$

We will use the following result by Karlin and Studden (1966), which we translated to semi-infinite LTP situations for coalition N.

Theorem 4.1. *Suppose that $v(N)$ is finite and that $\omega(N) \in \mathbb{R}^M_{++}$. Then there is no duality gap and the dual program has an optimal solution.*

We can now prove the following result.

Theorem 4.2. *Let $\langle N, A, D, \omega, p\rangle$ be a semi-infinite LTP situation. If Conditions 4.1 and 4.2 are satisfied then the corresponding LTP game is totally balanced.*

Proof. Since each subgame $(S, v_{|S})$ of an LTP game is another LTP game, we only have to prove that the core of (N, v) is non-empty.

By Conditions 4.1 and 4.2, it follows that the dual feasible region $\{z \in \mathbb{R}^M_+ \mid G^T z \geq A^T p\}$ is non-empty since $z^T = \gamma/\alpha(1, 1, \ldots, 1)$ is a feasible dual solution. It also follows that the primal profit maximisation problem has a finite optimal profit. From the result by Karlin and Studden (Theorem 4.1), it follows that if $\omega(N) \in \mathbb{R}^M_{++}$ then there is no duality gap and there exists an optimal dual solution $\underline{z}$. As we have shown in Theorem 3.1, the vector $x \in \mathbb{R}^N$ with $x_i = (\underline{z} + p)^T \omega(i)$ for all $i \in N$ is an element of $C(v)$.

If $\omega(N) \notin \mathbb{R}^M_{++}$ then one or more goods in M are not available, that is, there exists at least one good $j \in M$ such that $\omega_j(N) = 0$. We may eliminate these goods and all techniques that need a positive amount of them since it is impossible to use these transformation techniques. This reduced problem satisfies $\omega_j(N) > 0$ for all non-eliminated goods j. Again by the result of Karlin and Studden, it follows that there is no duality gap in this reduced problem, and there exists an optimal solution $\hat{z}$. To obtain an element of $C(v)$ we define $\underline{z}_j = \hat{z}_j$ for all non-eliminated goods j, and $\underline{z}_j = 0$ for all eliminated goods j. Then we can show in a similar way as in the proof of Theorem 3.1 that $x \in \mathbb{R}^N$, $x_i = (\underline{z} + p)^T \omega(i)$, is a core-element of the corresponding LTP game. $\square$

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